J. Math. Anal. Appl. 384 (2011) 578–590

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On G-symmetric solutions of a quasilinear elliptic equation involving critical Hardy–Sobolev exponent [☆]

Zhiying Deng [a,b,*], Yisheng Huang [a]

[a] Department of Mathematics, Soochow University, Suzhou 215006, Jiangsu, PR China
[b] School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China

ARTICLE INFO

Article history:
Received 19 January 2011
Available online 12 June 2011
Submitted by Goong Chen

Keywords:
G-symmetric solution
Critical Hardy–Sobolev exponent
Quasilinear elliptic equation

ABSTRACT

This paper deals with the class of singular quasilinear elliptic problem

$$-\Delta_p u = \mu \frac{|u|^{p-2}u}{|x|^p} + k(x)\frac{|u|^{p^*(s)-2}u}{|x|^s} + \lambda f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and Ω is G-symmetric with respect to a subgroup G of $O(\mathbb{N})$, $0 \in \Omega$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $0 \leqslant \mu < \bar{\mu}$, $\bar{\mu} = (\frac{N-p}{p})^p$, $0 \leqslant s < p$, $\lambda \geqslant 0$, $p^*(s) = \frac{(N-s)p}{N-p}$, $k(x)$ is continuous and G-symmetric on $\overline{\Omega}$, and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous nonlinearity of lower order satisfying some conditions. By using the variational methods and analytic techniques, we obtain several existence and multiplicity results of G-symmetric solutions under certain hypotheses on μ, λ and k.

1. Introduction

In recent years, many papers have been devoted to the study of the singular quasilinear elliptic problem of the type

$$\begin{cases} -\Delta_p u = \mu \dfrac{|u|^{p-2}u}{|x|^p} + f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain (bounded or unbounded) containing the origin, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $0 \leqslant \mu < \bar{\mu}$, $\bar{\mu} = (\frac{N-p}{p})^p$ is the best Hardy constant and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function with critical growth. This kind of equation arises in many areas of applied physics, including celestial mechanics, fluid mechanics, Newtonian fluids, and flow through porous media (see, for example [1]). Due to this fact, many existence, nonexistence and multiplicity results to equations like (1.1) have been obtained with various hypotheses on the measurable function $f(x,u)$, see for example [2–7] and the references therein.

Recently, Deng and Jin [8] considered the existence of nontrivial solutions of the following critical singular problem

$$-\Delta u = \mu \frac{u}{|x|^2} + k(x)\frac{u^{2^*(s)-1}}{|x|^s} \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

☆ Supported by the Natural Science Foundation of China (No. 11071180), and a grant (No. 08KJB110009) from the Jiangsu Education Committee of China.
* Corresponding author at: School of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China.
E-mail addresses: dengzy@cqupt.edu.cn (Z. Deng), yishengh@suda.edu.cn (Y. Huang).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.06.012

where $N > 2$, $0 \leqslant s < 2$, $0 \leqslant \mu < \bar{\mu} = (\frac{N-2}{2})^2$, $2^*(s) = \frac{2(N-s)}{N-2}$ is the critical Hardy–Sobolev exponent, and k is a bounded, continuous function satisfying some symmetry conditions with respect to a subgroup G of $O(\mathbb{N})$. By using a variant of the concentration–compactness principle of P.L. Lions together with the standard variational arguments, they proved the existence and multiplicity of G-symmetric solutions under some assumptions on k. Very recently, Deng and Huang [9] extended the results in [8] to the critical semilinear elliptic equations of Caffarelli–Kohn–Nirenberg type. We also mention that when $\mu = 0$, $s = 0$ and the right-hand side term $u^{2^*(s)-1}$ is replaced by u^{q-1} ($1 < q < \frac{2N}{N-2}$ or $q = \frac{2N}{N-2}$) in (1.2), the existence and multiplicity of G-symmetric solutions of (1.2) were obtained in [10–12].

In this paper, motivated by [8,10], we consider the following singular elliptic equation

$$\begin{cases} -\Delta_p u = \mu \dfrac{|u|^{p-2}u}{|x|^p} + k(x)\dfrac{|u|^{p^*(s)-2}u}{|x|^s} + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded, G-symmetric domain (see Section 2 for details) in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$, $0 \in \Omega$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $0 \leqslant \mu < \bar{\mu}$, $\bar{\mu} = (\frac{N-p}{p})^p$, $0 \leqslant s < p$, $\lambda \geqslant 0$, $p^*(s) = \frac{(N-s)p}{N-p}$ is the critical Hardy–Sobolev exponent and $p^*(0) = p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $k \in \mathcal{C}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$ and $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy some conditions which will be specified later. Due to the nonlinear perturbation $f(x,u)$ and the singularities caused by the terms $\frac{1}{|x|^p}$ and $\frac{1}{|x|^s}$, compared with the semilinear equation (1.2), the quasilinear equation (1.3) becomes more complicated to deal with and we have to overcome more difficulties in the study of G-symmetric solutions. To our knowledge, there is no result on the existence of G-symmetric solutions of (1.3) for $p \neq 2$, $s \neq 0$ and $\lambda \geqslant 0$. Note that, we have three parameters μ, s and λ here and we will try to treat the both cases of $\lambda = 0$ and $\lambda > 0$.

This paper is organized as follows. In Section 2 we will establish the appropriate Sobolev space which is applicable to the study of the problem (1.3), and will give the main results of this paper. In Section 3, we detail the proofs of some existence and multiplicity results for the case $\lambda = 0$ in (1.3). In Section 4, we give the proofs of existence and multiplicity results of G-symmetric solutions for the case $\lambda > 0$ in (1.3). Our methods in this paper are mainly based on a variant of the concentration–compactness principle of P.L. Lions and variational arguments.

2. Preliminaries and main results

Let $O(\mathbb{N})$ be the group of orthogonal linear transformations of $\mathbb{R}^N$ with natural action and let $G \subset O(\mathbb{N})$ be a subgroup with the property that $\mathrm{Fix}\{G\} = \{0\}$, where $\mathrm{Fix}\{G\} \triangleq \{x \in \mathbb{R}^N;\ gx = x,\ \forall g \in G\}$ is the fixed point set of the action of G on $\mathbb{R}^N$. For $x \neq 0$ we denote the cardinality of $G_x = \{gx;\ g \in G\}$ by $|G_x|$ and set

$$|G| \triangleq \inf\{|G_x|;\ x \in \mathbb{R}^N \backslash \{0\}\}.$$

In particular, $|G|$ may be $+\infty$. We call Ω a G-symmetric subset of $\mathbb{R}^N$, if $x \in \Omega$, then $gx \in \Omega$ for all $g \in G$. For any $f(x)$ defining on $\mathbb{R}^N$, we call $f(x)$ a G-symmetric function if for all $g \in G$ and $x \in \mathbb{R}^N$, $f(gx) = f(x)$ holds.

Let $W_0^{1,p}(\Omega)$ denote the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla \cdot|^p\,dx)^{1/p}$. We recall that the Hardy–Sobolev inequality (cf. [3]) asserts that for all $W_0^{1,p}(\Omega)$, there is a constant $C = C(q,s,p) > 0$ such that

$$\left(\int_\Omega |x|^{-s}|u|^q\,dx\right)^{p/q} \leqslant C\int_\Omega |\nabla u|^p\,dx, \quad p \leqslant q \leqslant p^*(s). \tag{2.1}$$

As $q = s = p$, the inequality (2.1) becomes the well-known Hardy inequality (cf. [2,3]),

$$\int_\Omega \frac{|u|^p}{|x|^p}\,dx \leqslant \frac{1}{\bar{\mu}}\int_\Omega |\nabla u|^p\,dx, \quad \forall u \in W_0^{1,p}(\Omega), \tag{2.2}$$

where $\bar{\mu} = (\frac{N-p}{p})^p$. Now we employ the following norm in $W_0^{1,p}(\Omega)$,

$$\|u\| \triangleq \left[\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx\right]^{1/p}, \quad 0 \leqslant \mu < \bar{\mu}.$$

By the Hardy inequality (2.2) we easily see that the above norm is equivalent to the norm $(\int_\Omega |\nabla u|^p\,dx)^{1/p}$.

For a bounded and G-symmetric domain $\Omega \subset \mathbb{R}^N$, $0 \in \Omega$, the natural functional space to study the problem (1.3) is the Banach space $W_{0,G}^{1,p}(\Omega)$ which is the subspace of $W_0^{1,p}(\Omega)$ consisting of all G-symmetric functions. Now in this paper, we consider the following problems

$$(\mathscr{P}_\lambda) \quad \begin{cases} -\Delta_p u = \mu \dfrac{|u|^{p-2}u}{|x|^p} + k(x)\dfrac{|u|^{p^*(s)-2}u}{|x|^s} + \lambda f(x,u), & x \in \Omega, \\ u > 0 \quad \text{in } \Omega, \quad \text{and} \quad u \in W_{0,G}^{1,p}(\Omega). \end{cases}$$

Before stating our results, we introduce two notations $\mathscr{A}_\mu$ and $y_\epsilon(x)$ which are respectively, defined by

$$\mathscr{A}_\mu \triangleq \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla u|^p - \mu \frac{|u|^p}{|x|^p})\,dx}{(\int_\Omega |x|^{-s}|u|^{p^*(s)}\,dx)^{p/p^*(s)}} \tag{2.3}$$

and

$$y_\epsilon(x) = C\epsilon^{-\delta} U_\mu\left(\frac{|x|}{\epsilon}\right), \tag{2.4}$$

where $\epsilon > 0$, $\delta \triangleq (N-p)/p$ and the constant $C = C(N, s, p, \mu) > 0$, depending only on N, s, p and μ. From [7], we know that $\mathscr{A}_\mu$ is independent of Ω and $y_\epsilon(x)$ satisfies the equations

$$\int_{\mathbb{R}^N} \left(|\nabla y_\epsilon|^p - \mu \frac{|y_\epsilon|^p}{|x|^p}\right) dx = 1 \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} |x|^{-s} y_\epsilon^{p^*(s)-1} \nu\,dx = \mathscr{A}_\mu^{-\frac{p^*(s)}{p}} \int_{\mathbb{R}^N} \left(|\nabla y_\epsilon|^{p-2}\nabla y_\epsilon \nabla \nu - \mu \frac{|y_\epsilon|^{p-2} y_\epsilon \nu}{|x|^p}\right) dx$$

for all $\nu \in \mathscr{D}^{1,p}(\mathbb{R}^N)$, where $\mathscr{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N);\ |\nabla u| \in L^p(\mathbb{R}^N)\}$. In particular, we have (let $\nu = y_\epsilon$)

$$\int_{\mathbb{R}^N} |x|^{-s} y_\epsilon^{p^*(s)}\,dx = \mathscr{A}_\mu^{-\frac{p^*(s)}{p}}. \tag{2.6}$$

$U_\mu > 0$ is radially symmetric and decreasing. Moreover, the following asymptotic properties at the origin and infinity for $U_\mu(r)$ and $U'_\mu(r)$ hold (cf. [7]):

$$\lim_{r\to 0} r^{l_1} U_\mu(r) = C_1 > 0, \tag{2.7}$$

$$\lim_{r\to +\infty} r^{l_2} U_\mu(r) = C_2 > 0, \tag{2.8}$$

$$\lim_{r\to 0} r^{l_1+1} U'_\mu(r) = C_1 l_1 > 0, \tag{2.9}$$

$$\lim_{r\to +\infty} r^{l_2+1} U'_\mu(r) = C_2 l_2 > 0, \tag{2.10}$$

where C_1, C_2 are positive constants depending only on N and p, and l_1, l_2 are the zeroes of the function

$$\mathscr{L}(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \geqslant 0,\ 0 \leqslant \mu < \bar{\mu} \tag{2.11}$$

satisfying

$$0 \leqslant l_1 < \delta < l_2 \leqslant \frac{N-p}{p-1}. \tag{2.12}$$

We suppose that $k(x)$ and $f(x,u)$ verify the following hypotheses.

(k.1) $k \in \mathcal{C}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, and $k(x)$ is G-symmetric.
(k.2) $k_+ \not\equiv 0$, where $k_+ = \max(0, k)$.
(k.3) There exists $\varsigma_0 > 0$ such that $k(x) \geqslant \varsigma_0$ on Ω.
(f.1) There exists $\tau \in (0,1)$ such that $f(x,u) \in \mathcal{C}^{0,\tau}(\overline{\Omega} \times [-M, M], \mathbb{R})$ for all $M > 0$.
(f.2) $f(x,0) = 0$, $\lim_{u\to 0} \frac{f(x,u)}{|x|^{-p}|u|^{p-1}} = 0$ uniformly in $x \in \Omega$.
(f.3) $\lim_{u\to\infty} \frac{f(x,u)}{|x|^{-s}|u|^{p^*(s)-1}} = 0$ uniformly in $x \in \Omega$.
(f.4) $f(x,u)$ is G-symmetric in $x \in \Omega$ for each $u \in \mathbb{R}$.
(f.5) There exist $m_0 > 0$, $0 < a < b < +\infty$ and a G-symmetric domain $\Omega_0 \subset \Omega$, with $0 \in \Omega_0$, such that (i) $\forall u \geqslant 0$, $f(x,u) \geqslant 0$ for a.e. $x \in \Omega_0$, and (ii) $\forall u \in (a,b)$, $f(x,u) \geqslant m_0 > 0$ for a.e. $x \in \Omega_0$.

The main results of this paper are the following.

Theorem 2.1. *Suppose that* (k.1) *and* (k.2) *hold. If*

$$\int_{\Omega} k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx \geqslant \max\left\{k_+(0),\left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-N}}|G|^{\frac{p-s}{p-N}}\|k_+\|_\infty\right\}\int_{\mathbb{R}^N}\frac{|y_\epsilon|^{p^*(s)}}{|x|^s}\,dx > 0 \tag{2.13}$$

for some $\epsilon > 0$*, where* $V_\epsilon = \phi y_\epsilon/\|\phi y_\epsilon\|$ *satisfies* (3.12)–(3.14)*, then Problem* $(\mathscr{P}_0)$ *has at least one positive solution in* $W^{1,p}_{0,G}(\Omega)$.

Corollary 2.1. *Suppose that* (k.1) *and* (k.2) *hold. Then Problem* $(\mathscr{P}_0)$ *has at least one positive solution in* $W^{1,p}_{0,G}(\Omega)$ *if*

$$k(0) > 0, \qquad k(0) \geqslant (\mathscr{A}_0/\mathscr{A}_\mu)^{\frac{N-s}{p-N}}|G|^{\frac{p-s}{p-N}}\|k_+\|_\infty \tag{2.14}$$

and $k(x) \geqslant k(0) + \gamma_0|x|^\vartheta$ *for some* $\gamma_0 > 0$, $\vartheta \in (0, p(l_2 - \delta))$ *and* $|x|$ *small.*

Theorem 2.2. *Suppose that* $k_+(0) = 0$ *and* $|G| = +\infty$*. Then Problem* $(\mathscr{P}_0)$ *has infinitely many* G*-symmetric solutions.*

Theorem 2.3. *Suppose that* (k.1), (k.3) *and* (f.1)–(f.5) *hold. Then there exists some* $\lambda_0 \geqslant 0$ *such that Problem* $(\mathscr{P}_\lambda)$ *possesses at least one positive solution in* $W^{1,p}_{0,G}(\Omega)$ *for every* $\lambda \geqslant \lambda_0$.

Remark 2.1. The existence and multiplicity results of $(\mathscr{P}_0)$ and $(\mathscr{P}_1)$ in the case $\mu = s = 0$, $p = 2$ have been obtained in [10]. This result has been extended to the case $p = 2$, $\lambda = 0$ and $\Omega = \mathbb{R}^N$ in [8]. Also, Abdellaoui et al. [6] obtained some existence and multiplicity results of $(\mathscr{P}_0)$ for the case $s = 0$ and $u \in \mathscr{D}^{1,p}(\mathbb{R}^N)$.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^N$ is G-symmetric. Denote by $W^{1,p}_{0,G}(\Omega)$ the subspace of $W^{1,p}_0(\Omega)$ consisting of all G-symmetric functions. The dual space of $W^{1,p}_{0,G}(\Omega)$ ($W^{1,p}_0(\Omega)$, resp.) is denoted by $W^{-1,p'}_G(\Omega)$ ($W^{-1,p'}(\Omega)$, resp.), where $1/p + 1/p' = 1$. The ball of center x and radius r is denoted by $B(x,r)$. Various positive constants whose exact values are meaningless are denoted by C. Denote by "$\to$" convergence in norm in a given Banach space X and by "$\rightharpoonup$" weak convergence. A functional $J \in \mathcal{C}^1(X,\mathbb{R})$ is said to satisfy the $(PS)_c$ condition if, each sequence $\{u_n\}$ in X satisfying $J(u_n) \to c$, $J'(u_n) \to 0$ in X^* has a subsequence which strongly converges to some element in X. Hereafter $L^q(\Omega, |x|^{-\alpha})$ denotes the weighted $L^q(\Omega)$ space with the norm $(\int_\Omega |x|^{-\alpha}|u|^q\,dx)^{1/q}$.

3. Existence and multiplicity results for Problem $(\mathscr{P}_0)$

We associate with Problem $(\mathscr{P}_0)$ a functional $J_0 : W^{1,p}_{0,G}(\Omega) \to \mathbb{R}$ given by

$$J_0(u) = \frac{1}{p}\int_\Omega\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - \frac{1}{p^*(s)}\int_\Omega k(x)\frac{|u|^{p^*(s)}}{|x|^s}\,dx. \tag{3.1}$$

Obviously, it follows from the inequalities (2.1) and (2.2) that the functional J_0 is well defined and of $\mathcal{C}^1$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of the problem and the critical points of J_0. More precisely, the (weak) solutions of Problem $(\mathscr{P}_0)$ are exactly the critical points of the functional J_0 by the following symmetric principle (see Lemma 3.1), namely $u \in W^{1,p}_{0,G}(\Omega)$ satisfies $(\mathscr{P}_0)$ if and only if for all $v \in W^{1,p}_0(\Omega)$

$$\int_\Omega\left(|\nabla u|^{p-2}\nabla u\nabla v - \mu\frac{|u|^{p-2}uv}{|x|^p}\right)dx - \int_\Omega k(x)\frac{|u|^{p^*(s)-2}uv}{|x|^s}\,dx = 0. \tag{3.2}$$

Lemma 3.1. *Let* $k(x)$ *be a* G*-symmetric function*; $J_0'(u) = 0$ *in* $W^{-1,p'}_G(\Omega)$ *implies* $J_0'(u) = 0$ *in* $W^{-1,p'}(\Omega)$.

Proof. For simplicity, we assume that $G \subset O(\mathbb{N})$ is closed. Since $O(\mathbb{N})$ is a compact Lie group, G is compact. Applying the principle of symmetric criticality (cf. [13, Theorem 5.4]), we obtain the conclusion. If G is not closed, we use an approximate argument as following. Without loss of generality, let $\overline{B(0,1)} \subset \Omega$ and $\varphi \in \mathcal{C}^\infty(\Omega)$ be such that $\operatorname{supp}\varphi \subset \overline{B(0,1)}$, $\int_\Omega \varphi(x)\,dx = 1$ and $\varphi(gx) = \varphi(x)$ for all $g \in G$. Then $\varphi_\epsilon(x) \triangleq \epsilon^{-N}\varphi(\frac{x}{\epsilon})$ is G-symmetric, and therefore $\overline{G}$-symmetric since φ_ϵ is continuous. Denote $\eta_\epsilon \triangleq \varphi_\epsilon * u$, where $*$ denotes the convolution. Since for all $g \in G$,

$$\begin{aligned}\eta_\epsilon(gx) &= \int_\Omega u(gx-y)\varphi_\epsilon(y)\,dy \overset{y=gz}{=} \int_\Omega u\big(g(x-z)\big)\varphi_\epsilon(gz)\,dz\\ &= \int_\Omega u(x-z)\varphi_\epsilon(z)\,dz = (\varphi_\epsilon * u)(x) = \eta_\epsilon(x),\end{aligned}$$

η_ϵ is also $\overline{G}$-symmetric by the continuity of η_ϵ. This, combined with the Rellich–Kondrachov type compactness result (cf. [4, Theorem 2.1]), implies that $\eta_\epsilon(x) \to u(x)$ in $L^q(\Omega, |x|^{-\alpha})$ for $1 \leqslant q < \frac{Np}{N-p}$, $\frac{\alpha}{q} < 1 + N(\frac{1}{q} - \frac{1}{p})$. Therefore we have that $\eta_\epsilon(x) \to u(x)$ a.e. in x, $\eta_\epsilon(gx) \to u(gx)$ a.e. in x for all $g \in \overline{G}$. On the other hand, $\eta_\epsilon(gx) = \eta_\epsilon(x)$, which means $u(gx) = u(x)$ and then u is $\overline{G}$-symmetric.

Alternately, we have $J_0'(gu) = J_0'(u)$ as in [14], that is $J_0'(u(gx)) = J_0'(u(x))$ a.e. in $x \in \Omega$. Set $v_1(x) \triangleq \int_G v(gx)\, d\mu_g$ for all $x \in \Omega$, where μ_g is the Haar measure. Since G is compact topological group, there exists a Haar measure μ_g such that $\mu_g(G) = 1$ (cf. [15]). Then $v_1 \in W_{0,G}^{1,p}(\Omega)$ since G acts by isometrics. Set $J_0'(u(x)) = w(x)$. Then $w \in W_G^{-1,p'}(\Omega)$, and we deduce from the Fubini theorem that

$$\begin{aligned}\langle w, v_1\rangle &= \int_\Omega \int_G w(gx)v(gx)\, d\mu_g\, dx = \int_G \int_\Omega w(gx)v(gx)\, dx\, d\mu_g \\ &= \int_G \int_\Omega w(x)v(x)\, dx\, d\mu_g = \int_G d\mu_g \langle w, v\rangle = \langle w, v\rangle = 0.\end{aligned}$$

Therefore the conclusion follows. $\square$

To establish conditions under which the Palais–Smale condition holds we need the following concentration–compactness principle due to Lions (cf. [16]).

Lemma 3.2. *Let $\{u_n\}$ be a weakly convergent sequence to u in $W_{0,G}^{1,p}(\Omega)$ such that $|\nabla u_n|^p \rightharpoonup \eta$, $|x|^{-s}|u_n|^{p^*(s)} \rightharpoonup \nu$ and $|x|^{-p}|u_n|^p \rightharpoonup \tilde{\nu}$ in the sense of measures. Then there exists some at most countable set $\mathscr{J}$, $\{\eta_j \geqslant 0\}_{j\in\mathscr{J}\cup\{0\}}$, $\{\nu_j \geqslant 0\}_{j\in\mathscr{J}\cup\{0\}}$, $\tilde{\nu}_0 \geqslant 0$, $\{x_j\}_{j\in\mathscr{J}} \subset \overline{\Omega}\backslash\{0\}$ such that*

(a) $\eta \geqslant |\nabla u|^p + \sum_{j\in\mathscr{J}} \eta_j \delta_{x_j} + \eta_0\delta_0$,
(b) $\nu = |x|^{-s}|u|^{p^*(s)} + \sum_{j\in\mathscr{J}} \nu_j \delta_{x_j} + \eta_0\delta_0$,
(c) $\tilde{\nu} = |x|^{-p}|u|^p + \tilde{\nu}_0\delta_0$,
(d) $\mathscr{A}_0 \nu_j^{p/p^*(s)} \leqslant \eta_j$,
(e) $\mathscr{A}_\mu \nu_0^{p/p^*(s)} \leqslant \eta_0 - \mu\tilde{\nu}_0$,

where δ_{x_j}, $j \in \mathscr{J} \cup \{0\}$, is the Dirac-mass of 1 *concentrated at $x_j \in \overline{\Omega}$.*

Proof. The proof is similar to that of the concentration–compactness principle in [16] and is omitted here. $\square$

To find critical points of J_0 we need the following local $(PS)_c$ condition which is crucial for the proof of Theorem 2.1.

Lemma 3.3. *Suppose that* (k.1) *and* (k.2) *hold. Then the $(PS)_c$ condition in $W_{0,G}^{1,p}(\Omega)$ holds for $J_0(u)$ if*

$$c < \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\mathscr{A}_\mu^{\frac{N-s}{p-s}} \min\left\{k_+(0)^{\frac{p-N}{p-s}}, |G|\left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-s}} \|k_+\|_\infty^{\frac{p-N}{p-s}}\right\}. \tag{3.3}$$

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for J_0 with c satisfying (3.3). Then by the inequality (2.1) we easily deduce that $\{u_n\}$ is bounded in $W_{0,G}^{1,p}(\Omega)$ and we may assume that $u_n \rightharpoonup u$ in $W_{0,G}^{1,p}(\Omega)$. By Lemma 3.2 there exist measures η, ν and $\tilde{\nu}$ such that relations (a)–(e) of this lemma hold. Let $x_j \neq 0$ be a singular point of measures η and ν. Since $x_j \neq 0$, $\forall j \in \mathscr{J}$, we can choose $\epsilon > 0$ small enough such that $0 \notin B(x_j, \epsilon)$ and $B(x_i, \epsilon) \cap B(x_j, \epsilon) = \emptyset$ for $i \neq j$, $i, j \in \mathscr{J}$. Let ϕ_ϵ be a smooth cut-off function center at x_j such that $0 \leqslant \phi_\epsilon \leqslant 1$, $\phi_\epsilon = 1$ for $|x - x_j| \leqslant \epsilon/2$, $\phi_\epsilon = 0$ for $|x - x_j| \geqslant \epsilon$, $|\nabla\phi_\epsilon| \leqslant 4/\epsilon$. By Lemma 3.1 we get

$$\begin{aligned}0 = \lim_{n\to\infty}\langle J_0'(u_n), u_n\phi_\epsilon\rangle = \lim_{n\to\infty}\Bigg\{&\int_\Omega \left(|\nabla u_n|^p\phi_\epsilon + u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\phi_\epsilon\right) dx \\ &- \mu\int_\Omega \frac{|u_n|^p}{|x|^p}\phi_\epsilon\, dx - \int_\Omega k(x)\frac{|u_n|^{p^*(s)}}{|x|^s}\phi_\epsilon\, dx\Bigg\}.\end{aligned} \tag{3.4}$$

Since $0 \notin \operatorname{supp}\phi_\epsilon$, we deduce from relations (a)–(c) of Lemma 3.2 that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\phi_\epsilon\,dx=\int_\Omega \phi_\epsilon\,d\eta\geqslant\int_\Omega |\nabla u|^p\phi_\epsilon\,dx+\eta_j, \tag{3.5}$$

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p^*(s)}}{|x|^s}\phi_\epsilon\,dx=\int_\Omega \phi_\epsilon\,d\nu=\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\phi_\epsilon\,dx+\nu_j, \tag{3.6}$$

$$\begin{aligned}\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi_\epsilon\,dx\right|&\leqslant\lim_{\epsilon\to0}\lim_{n\to\infty}\left(\int_\Omega |u_n|^p|\nabla\phi_\epsilon|^p\,dx\right)^{1/p}\left(\int_\Omega |\nabla u_n|^p\,dx\right)^{(p-1)/p}\\&\leqslant C\lim_{\epsilon\to0}\left(\int_\Omega |u|^p|\nabla\phi_\epsilon|^p\,dx\right)^{1/p}\\&\leqslant C\lim_{\epsilon\to0}\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{1/p^*}\left(\int_{B(x_j,\epsilon)} |\nabla\phi_\epsilon|^N\,dx\right)^{1/N}\\&\leqslant C\lim_{\epsilon\to0}\left(\int_{B(x_j,\epsilon)} |u|^{p^*}\,dx\right)^{1/p^*}=0,\end{aligned} \tag{3.7}$$

and

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_\Omega \frac{|u_n|^p\phi_\epsilon}{|x|^p}\,dx\right|\leqslant\lim_{\epsilon\to0}\lim_{n\to\infty}\left|\int_{B(x_j,\epsilon)} \frac{|u_n|^p\phi_\epsilon}{(|x_j|-\epsilon)^p}\,dx\right|=0. \tag{3.8}$$

Therefore (3.4)–(3.8) imply that

$$0=\lim_{\epsilon\to0}\lim_{n\to\infty}\langle J_0'(u_n),u_n\phi_\epsilon\rangle=\eta_j-\nu_j. \tag{3.9}$$

Obviously, if $k(x_j)\leqslant 0$ then $\eta_j=\nu_j=0$. Combining (3.9) and (d) of Lemma 3.2 we obtain that either

(i) $\nu_j=0$ or
(ii) $\nu_j\geqslant(\mathscr{A}_0/k(x_j))^{\frac{N-s}{p-s}}\geqslant(\mathscr{A}_0/\|k_+\|_\infty)^{\frac{N-s}{p-s}}$.

For the point $x=0$, similarly as in the case $x_j\neq0$, we get $\eta_0-\mu\tilde\nu_0-k(0)\nu_0\leqslant0$. This, combined with (e) of Lemma 3.2, implies that either

(iii) $\nu_0=0$ or
(iv) $\nu_0\geqslant(\mathscr{A}_\mu/k_+(0))^{\frac{N-s}{p-s}}$.

We now show that (ii) and (iv) cannot occur. For every continuous nonnegative function ψ such that $0\leqslant\psi(x)\leqslant1$ on Ω, we obtain

$$\begin{aligned}c&=\lim_{n\to\infty}\left[J_0(u_n)-\frac{1}{p^*(s)}\langle J_0'(u_n),u_n\rangle\right]=\left(\frac1p-\frac{1}{p^*(s)}\right)\lim_{n\to\infty}\int_\Omega\left(|\nabla u_n|^p-\mu\frac{|u_n|^p}{|x|^p}\right)dx\\&\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)\limsup_{n\to\infty}\int_\Omega\left(|\nabla u_n|^p-\mu\frac{|u_n|^p}{|x|^p}\right)\psi(x)\,dx.\end{aligned}$$

If (ii) occurs, then the set $\mathscr{J}$ must be finite because the measure ν is bounded. Since functions u_n are G-symmetric, the measure ν must be G-invariant. This means that if $x_j\neq0$ is a singular point of ν, so is gx_j for each $g\in G$ and the mass of ν concentrated at gx_j is the same for each $g\in G$. If we assume the existence of $j\in\mathscr{J}$ with $x_j\neq0$ such that (ii) holds, then we choose ψ with compact support so that $\psi(gx_j)=1$ for each $g\in G$ and we obtain

$$c\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)\eta_j|G|\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)\mathscr{A}_0|G|\nu_j^{\frac{p}{p^*(s)}}\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)|G|\mathscr{A}_0^{\frac{N-s}{p-s}}\|k_+\|_\infty^{\frac{p-N}{p-s}},$$

which contradicts (3.3). Similarly, if (iv) holds for $x=0$, we choose ψ with compact support, so that $\psi(0)=1$ and we obtain

$$c\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)(\eta_0-\mu\tilde\nu_0)\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)\mathscr{A}_\mu\nu_0^{\frac{p}{p^*(s)}}\geqslant\left(\frac1p-\frac{1}{p^*(s)}\right)\mathscr{A}_\mu^{\frac{N-s}{p-s}}k_+(0)^{\frac{p-N}{p-s}},$$

a contradiction with (3.3). Thus $\nu_j = 0$ for all $j \in \mathscr{J} \cup \{0\}$, and consequently we obtain $u_n \to u$ in $L^{p^*(s)}(\Omega, |x|^{-s})$. Finally, observe that $J_0'(u) = 0$ and, hence by $\lim_{n\to\infty}\langle J_0'(u_n) - J_0'(u), u_n - u\rangle = 0$ we get $u_n \to u$ in $W_0^{1,p}(\Omega)$. $\square$

As an immediate consequence of Lemma 3.3 we obtain the following result.

Corollary 3.1. *If $k_+(0) = 0$ and $|G| = +\infty$, then the functional J_0 satisfies $(PS)_c$ condition for every $c \in \mathbb{R}$.*

Proof of Theorem 2.1. Let $y_\epsilon(x)$ be the extremal function satisfying (2.5) and (2.6). First, we choose $\epsilon > 0$ such that the assumption (2.13) holds, where $V_\epsilon = \phi y_\epsilon / \|\phi y_\epsilon\|$ satisfies (3.12)–(3.14). By (2.1)–(2.3) we obtain

$$J_0(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x^p}\right)dx - \frac{1}{p^*(s)}\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\,dx \geqslant \frac{1}{p}\|u\|^p - \frac{\mathscr{A}_\mu^{-\frac{p^*(s)}{p}}}{p^*(s)}\|u\|^{p^*(s)}.$$

Consequently, there exist constants $\alpha_0 > 0$ and $\rho > 0$ such that $J_0(u) \geqslant \alpha_0$ for all $\|u\| = \rho$. On the other hand, if we set

$$\Phi(t) = J_0(tV_\epsilon) = \frac{t^p}{p}\int_\Omega \left(|\nabla V_\epsilon|^p - \mu\frac{|V_\epsilon|^p}{|x^p}\right)dx - \frac{t^{p^*(s)}}{p^*(s)}\int_\Omega k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx$$

with $t \geqslant 0$, then we can see easily that Φ has a unique maximum in positive t at some $\bar t > 0$. Simple arithmetic give us the value

$$\max_{t\geqslant 0}\Phi(t) = J_0(\bar t V_\epsilon) = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\left\{\frac{\int_\Omega(|\nabla V_\epsilon|^p - \mu\frac{|V_\epsilon|^p}{|x|^p})\,dx}{(\int_\Omega k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx)^{\frac{p}{p^*(s)}}}\right\}^{\frac{p^*(s)}{p^*(s)-p}}. \tag{3.10}$$

We now choose $t_0 > 0$ such that $J_0(t_0V_\epsilon) < 0$ and $\|t_0 V_\epsilon\| > \rho$ and set

$$c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_0\big(\gamma(t)\big),$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], W_{0,G}^{1,p}(\Omega));\ \gamma(0) = 0,\ J_0(\gamma(1)) < 0,\ \|\gamma(1)\| > \rho\}$. Setting

$$c_0^* = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\mathscr{A}_\mu^{\frac{N-s}{p-s}}\min\left\{k_+(0)^{\frac{p-N}{p-s}}, |G|\left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-s}}\|k_+\|_\infty^{\frac{p-N}{p-s}}\right\}, \tag{3.11}$$

we deduce from (2.3), (2.13), (3.10) and (3.11) that

$$\begin{aligned}
c_0 &\leqslant J_0(\bar t V_\epsilon) = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\left\{\frac{\int_\Omega(|\nabla V_\epsilon|^p - \mu\frac{|V_\epsilon|^p}{|x|^p})\,dx}{(\int_\Omega k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx)^{\frac{p}{p^*(s)}}}\right\}^{\frac{p^*(s)}{p^*(s)-p}} \\
&\leqslant \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\left\{\frac{\int_\Omega(|\nabla V_\epsilon|^p - \mu\frac{|V_\epsilon|^p}{|x|^p})\,dx}{(\max\{k_+(0), (\frac{\mathscr{A}_0}{\mathscr{A}_\mu})^{\frac{N-s}{p-N}}|G|^{\frac{p-s}{p-N}}\|k_+\|_\infty\}\int_{\mathbb{R}^N}\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx)^{\frac{p}{p^*(s)}}}\right\}^{\frac{p^*(s)}{p^*(s)-p}} \\
&= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\mathscr{A}_\mu^{\frac{N-s}{p-s}}\min\left\{k_+(0)^{\frac{p-N}{p-s}}, |G|\left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-s}}\|k_+\|_\infty^{\frac{p-N}{p-s}}\right\} = c_0^*.
\end{aligned}$$

If $c_0 < c_0^*$, then by Lemma 3.3, the $(PS)_c$ condition holds and the conclusion follows from the mountain pass lemma (see [17]). If $c_0 = c_0^*$, then $\gamma(t) = tt_0V_\epsilon$, with $0 \leqslant t \leqslant 1$, is a path in Γ such that $\max_{t\in[0,1]} J_0(\gamma(t)) = c_0$. Therefore, either $\Phi'(\bar t) = J_0'(\bar t V_\epsilon) = 0$ and we are done, or γ can be deformed to a path $\tilde\gamma \in \Gamma$ with $\max_{t\in[0,1]} J_0(\tilde\gamma(t)) < c_0$ and we get a contradiction. This part of the proof shows that a nontrivial solution $u_0 \in W_{0,G}^{1,p}(\Omega)$ of Problem $(\mathscr{P}_0)$ exists.

We now show that the solution u_0 can be chosen to be positive on Ω. Let the Nehari manifold M_0 associated to J_0 be defined by

$$\begin{aligned}
M_0 &\triangleq \big\{u \in W_{0,G}^{1,p}(\Omega)\backslash\{0\};\ \big\langle J_0'(u), u\big\rangle = 0\big\} \\
&= \left\{u \in W_{0,G}^{1,p}(\Omega)\backslash\{0\};\ \int_\Omega\left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - \int_\Omega k(x)\frac{|u|^{p^*(s)}}{|x|^s}\,dx = 0\right\}.
\end{aligned}$$

Writing an arbitrary element $u \in M_0$ as $u = t\nu$ $(t > 0)$, with $\|\nu\| = 1$, we deduce from (3.2) and the fact $\langle J_0'(t\nu), t\nu\rangle = 0$ that

$$0 = 1 - t^{p^*(s)-p} \int_\Omega k(x)\frac{|\nu|^{p^*(s)}}{|x|^s}\,dx \geqslant 1 - Ct^{p^*(s)-p},$$

which implies that $t \geqslant \sigma_0$, with a constant $\sigma_0 > 0$ independent of u. Therefore we conclude that the set M_0 is bounded away from 0 and $\inf_{u\in M_0} J_0(u) > 0$. Let $g(u) = \langle J_0'(u), u\rangle$, then $\langle g'(u), u\rangle = (p - p^*(s))\|u\|^p < 0$, and consequently, M_0 is a $\mathcal{C}^1$-manifold. Note that $u_0 \in M_0$ and set $\tilde{c}_0 = \inf_{u\in M_0} J_0(u)$. We now claim that $\tilde{c}_0 = c_0$. Arguing by contradiction, assume that $\tilde{c}_0 < c_0$; then we can find $u_1 \in M_0$ such that $J_0(u_1) < c_0$. Since $u_1 \in M_0$, we obtain $\int_\Omega k(x)|x|^{-s}|u_1|^{p^*(s)}\,dx = \|u_1\|^p > 0$. By a straightforward calculation, we get

$$\sup_{t\geqslant 0} J_0(tu_1) = J_0(u_1) = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\int_\Omega k(x)\frac{|u_1|^{p^*(s)}}{|x|^s}\,dx < C.$$

Setting $\gamma(t) = tt_1u_1$ for $t \in [0,1]$, with t_1 such that $\|t_1u_1\| > \rho$, we see that $J_0(\gamma(t)) = J_0(tt_1u_1) < c_0$ on $[0,1]$, which is impossible. Thus we have $\tilde{c}_0 = c_0$, and consequently by the strong maximum principle we obtain $u_0 > 0$ on Ω. $\square$

Proof of Corollary 2.1. Let $y_\epsilon(x)$ be the extremal function satisfying (2.4)–(2.12). Choose $\phi \in \mathcal{C}_0^1(\Omega)$ so that $\phi \geqslant 0$ on Ω and $\phi(x) = 1$ on $B(0,\varrho)$, with $\varrho > 0$ to be determined. We deduce from [7, Lemma 2.4] (see (2.21), (2.22) there), that

$$\|\phi y_\epsilon\|^p = \int_\Omega \left(\left|\nabla(\phi y_\epsilon)\right|^p - \mu\frac{|\phi y_\epsilon|^p}{|x|^p}\right)dx = 1 + O\left(\epsilon^{p(l_2-\delta)}\right) \tag{3.12}$$

and

$$\int_\Omega \frac{|\phi y_\epsilon|^{p^*(s)}}{|x|^s}\,dx = \mathscr{A}_\mu^{\frac{N-s}{p-N}} + O\left(\epsilon^{(l_2-\delta)p^*(s)}\right). \tag{3.13}$$

Set $V_\epsilon \triangleq \phi y_\epsilon/\|\phi y_\epsilon\|$, then by (3.12) and (3.13) we have

$$\begin{aligned}\int_\Omega \frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx &= \|\phi y_\epsilon\|^{-p^*(s)}\int_\Omega \frac{|\phi y_\epsilon|^{p^*(s)}}{|x|^s}\,dx\\ &= \left(1 + O\left(\epsilon^{p(l_2-\delta)}\right)\right)^{-\frac{p^*(s)}{p}}\left(\mathscr{A}_\mu^{\frac{N-s}{p-N}} + O\left(\epsilon^{(l_2-\delta)p^*(s)}\right)\right)\\ &= \mathscr{A}_\mu^{\frac{N-s}{p-N}} + O\left(\epsilon^{p(l_2-\delta)}\right).\end{aligned} \tag{3.14}$$

Let us now choose $\varrho > 0$ so that $k(x) \geqslant k(0) + \gamma_0|x|^\vartheta$ for $|x| \leqslant \varrho$. Then we deduce from (3.14) that

$$\int_\Omega k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx = \int_\Omega \left(k(x) - k(0)\right)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx + k(0)\mathscr{A}_\mu^{\frac{N-s}{p-N}} + O\left(\epsilon^{p(l_2-\delta)}\right).$$

It is sufficient to show that

$$\int_\Omega \left(k(x) - k(0)\right)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx + O\left(\epsilon^{p(l_2-\delta)}\right) \geqslant 0 \tag{3.15}$$

for sufficiently small $\epsilon > 0$. We have

$$\begin{aligned}\int_\Omega \left(k(x) - k(0)\right)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx &= \int_{|x|\leqslant\varrho} \left(k(x) - k(0)\right)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx + \int_{|x|\geqslant\varrho} \left(k(x) - k(0)\right)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx\\ &\geqslant \gamma_0 \int_{|x|\leqslant\varrho} \frac{|x|^\vartheta|y_\epsilon|^{p^*(s)}}{|x|^s\|\phi y_\epsilon(x)\|^{p^*(s)}}\,dx + \int_{|x|\geqslant\varrho} \frac{(k(x) - k(0))|\phi y_\epsilon|^{p^*(s)}}{|x|^s\|\phi y_\epsilon(x)\|^{p^*(s)}}\,dx = I_1 + I_2.\end{aligned}$$

For $\epsilon > 0$ sufficiently small, we deduce from (2.4)–(2.12), (3.12) and the fact $N - 1 + \vartheta - s - l_1p^*(s) > -1$, $N - 1 + \vartheta - s - l_2p^*(s) < -1$ that

$$
\begin{aligned}
I_1 &= \gamma_0 \int\limits_{|x|\leqslant\varrho} \frac{|x|^{\vartheta}|y_\epsilon|^{p^*(s)}}{|x|^s\|\phi y_\epsilon(x)\|^{p^*(s)}}\,dx \\
&= \gamma_0\big(1+O\big(\epsilon^{p(l_2-\delta)}\big)\big)^{-\frac{p^*(s)}{p}} \int\limits_{|x|\leqslant\varrho} |x|^{\vartheta-s}\left[C\epsilon^{-\delta}U_\mu\left(\frac{|x|}{\epsilon}\right)\right]^{p^*(s)} dx \\
&= \frac{\gamma_0 C^{p^*(s)}\epsilon^{-\delta p^*(s)}}{(1+O(\epsilon^{p(l_2-\delta)}))^{\frac{p^*(s)}{p}}} \int\limits_{|x|\leqslant\varrho} |x|^{\vartheta-s}\left[U_\mu\left(\frac{|x|}{\epsilon}\right)\right]^{p^*(s)} dx \\
&= \frac{C\epsilon^{\vartheta}}{(1+O(\epsilon^{p(l_2-\delta)}))^{\frac{p^*(s)}{p}}} \int\limits_{\frac{|x|}{\epsilon}\leqslant\frac{\varrho}{\epsilon}} \left(\frac{|x|}{\epsilon}\right)^{\vartheta-s-l_2p^*(s)}\left[\left(\frac{|x|}{\epsilon}\right)^{l_2}U_\mu\left(\frac{|x|}{\epsilon}\right)\right]^{p^*(s)} d\left(\frac{x}{\epsilon}\right) \\
&\geqslant C\epsilon^{\vartheta}\left\{\int\limits_{|x|\leqslant 1}\frac{(|x|^{l_1}U_\mu(|x|))^{p^*(s)}}{|x|^{-\vartheta+s+l_1p^*(s)}}\,dx + \int\limits_{1<|x|\leqslant\frac{\varrho}{\epsilon}}\frac{(|x|^{l_2}U_\mu(|x|))^{p^*(s)}}{|x|^{-\vartheta+s+l_2p^*(s)}}\,dx\right\} \\
&\geqslant \bar{C}_1\epsilon^{\vartheta}, \quad \vartheta\in\big(0, p(l_2-\delta)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\leqslant \int\limits_{|x|\geqslant\varrho} \frac{|k(x)-k(0)||\phi y_\epsilon|^{p^*(s)}}{|x|^s\|\phi y_\epsilon(x)\|^{p^*(s)}}\,dx \leqslant C\epsilon^{-\delta p^*(s)} \int\limits_{|x|\geqslant\varrho} |x|^{-s}\left[U_\mu\left(\frac{|x|}{\epsilon}\right)\right]^{p^*(s)} dx \\
&= C \int\limits_{\frac{|x|}{\epsilon}\geqslant\frac{\varrho}{\epsilon}} \left(\frac{|x|}{\epsilon}\right)^{-s-l_2p^*(s)}\left[\left(\frac{|x|}{\epsilon}\right)^{l_2}U_\mu(\frac{|x|}{\epsilon})\right]^{p^*(s)} d\left(\frac{x}{\epsilon}\right) \\
&\leqslant C\int_{\varrho\epsilon^{-1}}^{+\infty} r^{N-1-s-l_2p^*(s)}\,dr \leqslant \bar{C}_2\epsilon^{(l_2-\delta)p^*(s)},
\end{aligned}
$$

where $\bar{C}_1>0$ and $\bar{C}_2>0$ are constants independent of ϵ. Since $0<\vartheta<p(l_2-\delta)<(l_2-\delta)p^*(s)$, the inequality (3.15) follows as $\epsilon>0$ is small enough. Therefore we conclude from (2.6), (2.14) and (3.15) that

$$
\begin{aligned}
\int\limits_{\Omega} k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx &= \int\limits_{\Omega}\big(k(x)-k(0)\big)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx + \int\limits_{\Omega} k(0)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx \\
&\geqslant k(0)\int\limits_{\mathbb{R}^N}\frac{|y_\epsilon|^{p^*(s)}}{|x|^s}\,dx \geqslant \max\left\{k_+(0), \left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-N}}|G|^{\frac{p-s}{p-N}}\|k_+\|_\infty\right\}\int\limits_{\mathbb{R}^N}\frac{|y_\epsilon|^{p^*(s)}}{|x|^s}\,dx > 0.
\end{aligned}
$$

By the above inequality and the same process as in the proof of Theorem 2.1, we obtain the conclusion. □

To prove Theorem 2.2 we need the following version of symmetric mountain pass lemma (cf. [18, Theorem 9.12]).

Lemma 3.4. *Let E be an infinite dimensional Banach space and let $J\in\mathcal{C}^1(E,\mathbb{R})$ be an even functional satisfying $(PS)_c$ condition for each c and $J(0)=0$. Further, we suppose that:*

(i) *there exist constants $\alpha>0$ and $\rho>0$ such that $J(u)\geqslant\alpha$ for all $\|u\|=\rho$;*
(ii) *there exists an increasing sequence of subspaces $\{E_m\}$ of E, with $\dim E_m=m$, such that for every m one can find a constant $R_m>0$ such that $J(u)\leqslant 0$ for all $u\in E_m$ with $\|u\|\geqslant R_m$.*

Then J possesses a sequence of critical values $\{c_m\}$ tending to ∞ as $m\to\infty$.

Proof of Theorem 2.2. We follow the arguments of [10]. Now we apply Lemma 3.4 with $E=W^{1,p}_{0,G}(\Omega)$. By (2.1), (2.3) and (3.1) we deduce that

$$\begin{aligned} J_0(u) &= \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - \frac{1}{p^*(s)}\int_\Omega k(x)\frac{|u|^{p^*(s)}}{|x|^s}\,dx \\ &\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\|k\|_\infty \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\,dx \\ &\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\|k\|_\infty \mathscr{A}_\mu^{-\frac{p^*(s)}{p}}\|u\|^{p^*(s)}. \end{aligned}$$

Since $p^*(s) > p$, there exist $\bar{\alpha} > 0$ and $\rho > 0$ such that $J(u) \geqslant \bar{\alpha}$ for all u with $\|u\| = \rho$. To find a suitable sequence of finite dimensional subspaces of $W^{1,p}_{0,G}(\Omega)$, we set $\Omega_k = \{x \in \Omega;\ k(x) > 0\}$. Obviously, the set Ω_k is G-symmetric and we can define $W^{1,p}_{0,G}(\Omega_k)$, which is the subspace of G-symmetric functions of $W^{1,p}_0(\Omega_k)$ (see Section 2). By extending functions in $W^{1,p}_{0,G}(\Omega_k)$ to 0 outside Ω_k we can assume that $W^{1,p}_{0,G}(\Omega_k) \subset W^{1,p}_{0,G}(\Omega)$. Let $\{E_m\}$ be an increasing sequence of subspaces of $W^{1,p}_{0,G}(\Omega_k)$ with $\dim E_m = m$ for each m. Then we easily see that there exists a constant $\epsilon = \epsilon(m) > 0$ such that $\int_{\Omega_k} k(x)|x|^{-s}|\nu|^{p^*(s)}\,dx \geqslant \epsilon$ for all $\nu \in E_m$, with $\|\nu\| = 1$. Therefore, if $0 \neq u \in E_m$ then we write $u = t\nu$, with $t = \|u\|$ and $\|\nu\| = 1$. Hence we obtain

$$J_0(u) = \frac{1}{p}t^p - \frac{1}{p^*(s)}t^{p^*(s)}\int_{\Omega_k} k(x)\frac{|\nu|^{p^*(s)}}{|x|^s}\,dx \leqslant \frac{1}{p}t^p - \frac{\epsilon}{p^*(s)}t^{p^*(s)} \leqslant 0$$

for t large enough. Therefore we conclude from Corollary 3.1 and Lemma 3.4 that there exists a sequence of critical values $c_m \to \infty$ as $m \to \infty$ and the results follow. $\square$

4. Existence result for Problem $(\mathscr{P}_\lambda)$

The aim of this section is to discuss Problem $(\mathscr{P}_\lambda)$ and prove Theorem 2.3. We now define a functional $J_\lambda : W^{1,p}_{0,G}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx - \frac{1}{p^*(s)}\int_\Omega k(x)\frac{|u|^{p^*(s)}}{|x|^s}\,dx - \lambda\int_\Omega F(x,u)\,dx, \tag{4.1}$$

where $F(x,u) = \int_0^u f(x,t)\,dt$. Obviously, J_λ is well defined and of C^1, then there exists a one-to-one correspondence between the weak solutions of the problem and the critical points of J_λ. Moreover, an analogously symmetric principle of Lemma 3.1 clearly holds; hence the (weak) solutions of Problem $(\mathscr{P}_\lambda)$ with $\lambda > 0$ are exactly the critical points of the functional J_λ.

Lemma 4.1. *Suppose that* (k.1), (k.3) *and* (f.1)–(f.5) *hold. Then the following geometric conditions for* $J_\lambda(u)$ *hold*:

(i) $J_\lambda(0) = 0$, *there exist constants* $\tilde{\alpha} > 0$ *and* $\rho > 0$ *such that* $J_\lambda(u) \geqslant \tilde{\alpha}$ *for all* $\|u\| = \rho$;
(ii) *there exists* $e \in W^{1,p}_{0,G}(\Omega)$ *such that* $\|e\| > \rho$ *and* $J_\lambda(e) \leqslant 0$.

Proof. According to (f.1)–(f.3), for all $\epsilon > 0$, there exists $C_1(\epsilon) > 0$ such that

$$\left|F(x,u)\right| \leqslant \epsilon\frac{|u|^p}{|x|^p} + C_1(\epsilon)\frac{|u|^{p^*(s)}}{|x|^s}, \quad \forall (x,u) \in \Omega \times \mathbb{R}. \tag{4.2}$$

Consequently, we deduce from (2.1), (2.2) and (4.2) that

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega k(x)\frac{|u|^{p^*(s)}}{|x|^s}\,dx - \lambda\epsilon\int_\Omega \frac{|u|^p}{|x|^p}\,dx - \lambda C_1(\epsilon)\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\,dx \\ &\geqslant \left(\frac{1}{p} - \lambda C\epsilon\right)\|u\|^p - C\|u\|^{p^*(s)}, \end{aligned}$$

which implies (i) for $\epsilon > 0$ sufficiently small. We now choose $\tilde{u} \in W^{1,p}_{0,G}(\Omega)$ such that $\operatorname{supp}\tilde{u} \subset \Omega_0$, $\tilde{u} \not\equiv 0$. Then (k.3), (f.5) and (4.1) imply that

$$J_\lambda(t\tilde{u}) \leqslant \frac{1}{p}t^p\|\tilde{u}\|^p - \frac{\varsigma_0}{p^*(s)}t^{p^*(s)}\int_{\Omega_0} k(x)\frac{|\tilde{u}|^{p^*(s)}}{|x|^s}\,dx \to -\infty \quad \text{as } t \to +\infty,$$

and thus (ii) follows. $\square$

We set

$$c_\lambda \triangleq \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda\big(\gamma(t)\big)$$

with $\Gamma=\{\gamma\in\mathcal{C}([0,1],W^{1,p}_{0,G}(\Omega));\ \gamma(0)=0,\ J_\lambda(\gamma(1))<0,\ \|\gamma(1)\|>\rho\}$, and deduce from Lemma 4.1 and the mountain pass theorem without the (PS) condition (cf. [17]) the following result.

Lemma 4.2. *Suppose that* (k.1), (k.3) *and* (f.1)–(f.5) *hold. Then there exists a sequence* $\{u_n\}\subset W^{1,p}_{0,G}(\Omega)$ *such that* $J_\lambda(u_n)\to c_\lambda$ *and* $J'_\lambda(u_n)\to 0$ *in* $W^{-1,p'}_G(\Omega)$.

Lemma 4.3. *Suppose that* (k.1), (k.3) *and* (f.1)–(f.5) *hold. Then Problem* $(\mathscr{P}_\lambda)$ *possesses at least one positive solution in* $W^{1,p}_{0,G}(\Omega)$ *if*

$$c_\lambda < c^*_\lambda=\left(\frac{1}{p}-\frac{1}{p^*(s)}\right)\mathscr{A}_\mu^{\frac{N-s}{p-s}}\min\left\{k(0)^{\frac{p-N}{p-s}},\ |G|\left(\frac{\mathscr{A}_0}{\mathscr{A}_\mu}\right)^{\frac{N-s}{p-s}}\|k\|_\infty^{\frac{p-N}{p-s}}\right\}. \tag{4.3}$$

Proof. To get the existence of a positive solution we modify the nonlinearity by defining $\tilde f(x,u)=f(x,u)$ if $u\geqslant 0$ and $\tilde f(x,u)=0$ if $u<0$ and we set

$$\widetilde J_\lambda(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p-\mu\frac{|u|^p}{|x|^p}\right)dx-\frac{1}{p^*(s)}\int_\Omega k(x)\frac{|u^+|^{p^*(s)}}{|x|^s}\,dx-\lambda\int_\Omega \widetilde F(x,u)\,dx,$$

where $\widetilde F(x,u)=\int_0^u\tilde f(x,t)\,dt$, $u^+=\max\{u,0\}$. By virtue of Lemma 4.2 there exists a sequence $\{u_n\}\subset W^{1,p}_{0,G}(\Omega)$ such that $\widetilde J_\lambda(u_n)\to c_\lambda$ and $\widetilde J'_\lambda(u_n)\to 0$ as $n\to\infty$. On the other hand, we easily check from (k.3) that there exist constants $q_0=q_0(\lambda)\in(p,p^*(s))$ and $r>0$ such that

$$\frac{k(x)}{p^*(s)}|x|^{-s}|u|^{p^*(s)}+\lambda\widetilde F(x,u)\leqslant\frac{1}{q_0}\big[k(x)|x|^{-s}|u|^{p^*(s)}+\lambda u\tilde f(x,u)\big]$$

for all $x\in\Omega$ and $\|u\|\geqslant r$. Therefore there exist $C=C(q_0,\lambda,r)>0$ and $n_0\geqslant 1$ such that for $n\geqslant n_0$, we have

$$\begin{aligned}
c_\lambda+1&\geqslant\widetilde J_\lambda(u)-\frac{1}{q_0}\big\langle\widetilde J'_\lambda(u_n),u_n\big\rangle+\frac{1}{q_0}\big\langle\widetilde J'_\lambda(u_n),u_n\big\rangle\\
&\geqslant\left(\frac1p-\frac{1}{q_0}\right)\|u_n\|^p+\frac{1}{q_0}\int_\Omega\left(k(x)\frac{|u_n^+|^{p^*(s)}}{|x|^s}+\lambda u_n\tilde f(x,u_n)\right)dx\\
&\quad-\int_\Omega\left(\frac{k(x)}{p^*(s)}|x|^{-s}\big|u_n^+\big|^{p^*(s)}+\lambda\widetilde F(x,u_n)\right)dx+o(1)\|u_n\|\\
&\geqslant\left(\frac1p-\frac{1}{q_0}\right)\|u_n\|^p-C+o(1)\|u_n\|.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in $W^{1,p}_{0,G}(\Omega)$. We may, therefore, assume that $u_n\rightharpoonup u$ in $W^{1,p}_{0,G}(\Omega)$ and in $L^{p^*(s)}(\Omega,|x|^{-s})$, moreover, $u_n\to u$ in $L^q(\Omega,|x|^{-\alpha})$ for all $1\leqslant q<\frac{Np}{N-p}$, $\frac{\alpha}{q}<1+N(\frac1q-\frac1p)$ and a.e. on Ω. According to Lemma 3.2, there exist bounded measures η, ν and $\tilde\nu$ such that $|\nabla u_n|^p\rightharpoonup\eta$, $|x|^{-s}|u_n|^{p^*(s)}\rightharpoonup\nu$ and $|x|^{-p}|u_n|^p\rightharpoonup\tilde\nu$ in the sense of measures. Singular points of η, ν and $\tilde\nu$ are in $\overline{\Omega}$ and the corresponding masses η_j, ν_j and $\tilde\nu_0$ of η, ν and $\tilde\nu$, respectively, satisfy (a)–(e) of Lemma 3.2. Since $\langle\widetilde J'_\lambda(u_n),u_n^-\rangle=\|u_n^-\|^p\to 0$, where $u^-=\min(u,0)$, we get $u_n^-\to 0$ in $W^{1,p}_{0,G}(\Omega)$. Then we have $|x|^{-s}(u_n^+)^{p^*(s)}\rightharpoonup\nu$. It now suffices to prove that $u\not\equiv 0$ on Ω. The same process as in Lemma 3.3 shows that concentration can only occur at point x_j where $k(x_j)>0$. Assuming that $u\equiv 0$ on Ω, we obtain

$$\begin{aligned}
c_\lambda&=\lim_{n\to\infty}\left(\widetilde J_\lambda(u_n)-\frac1p\big\langle\widetilde J'_\lambda(u_n,u_n)\big\rangle\right)\\
&=\lim_{n\to\infty}\left\{\left(\frac1p-\frac{1}{p^*(s)}\right)\int_\Omega k(x)\frac{|u_n^+|^{p^*(s)}}{|x|^s}+\lambda\int_\Omega\left(\frac1p\tilde f(x,u_n)u_n-\widetilde F(x,u_n)\right)dx\right\}\\
&=\left(\frac1p-\frac{1}{p^*(s)}\right)\Big(\sum k(x_j)\nu_j+k(0)\nu_0\Big),
\end{aligned}$$

repeating the argument of Lemma 3.3 we easily deduce a contradiction with the assumption that $c_\lambda < c_\lambda^*$. The lemma is proved. □

Proof of Theorem 2.3. Applying Lemma 4.3, here we directly verify condition (4.3). Recall that $V_\epsilon(x) = \phi y_\epsilon / \|\phi y_\epsilon\|$ which satisfies (3.12)–(3.14). We fix $V_{\epsilon_0}(x)$ with $\epsilon_0 \in (0,1)$, $B(0,\varrho) \subset \Omega_0$, $\phi \in C_0^\infty(\Omega_0)$ and $\phi(0) = 1$. Set

$$\Psi_\lambda(t) = J_\lambda(tV_{\epsilon_0}) = \frac{1}{p}t^p - \frac{1}{p^*(s)}t^{p^*(s)} \int_\Omega k(x)\frac{|V_\epsilon|^{p^*(s)}}{|x|^s}\,dx - \lambda \int_\Omega F(x, tV_{\epsilon_0})\,dx, \quad t \geqslant 0.$$

We claim that

$$\lim_{\lambda\to+\infty} \sup_{t\geqslant 0} \Psi_\lambda(t) = 0, \tag{4.4}$$

and therefore condition (4.3) is satisfied when λ is large enough. First we obtain from (k.3), (f.5) that $\lim_{t\to+\infty} \Psi_\lambda(t) = -\infty$, and consequently $\sup_{t\geqslant 0} \Psi_\lambda(t)$ can be achieved at some t_λ for which we have

$$t_\lambda^{p-1} - t_\lambda^{p^*(s)-1} \int_\Omega k(x)\frac{|V_{\epsilon_0}|^{p^*(s)}}{|x|^s}\,dx - \lambda \int_\Omega f(x, t_\lambda V_{\epsilon_0})V_{\epsilon_0}\,dx = 0. \tag{4.5}$$

Therefore we obtain

$$0 \leqslant t_\lambda \leqslant \left(\int_\Omega k(x)\frac{|V_{\epsilon_0}|^{p^*(s)}}{|x|^s}\right)^{\frac{1}{p-p^*(s)}}.$$

It follows that

$$\lim_{\lambda\to+\infty} t_\lambda = 0. \tag{4.6}$$

Indeed, if not we could find some sequence $t_{\lambda_i} \to \xi > 0$ with $\lambda_i \to \infty$, then by (4.5) and the Fatou theorem we would have

$$\begin{aligned} 0 &\leqslant \int_\Omega f(x, \xi V_{\epsilon_0})V_{\epsilon_0}\,dx = \int_\Omega \lim_{\lambda_i\to\infty} f(x, t_{\lambda_i}V_{\epsilon_0})V_{\epsilon_0}\,dx \leqslant \lim_{\lambda_i\to\infty} \int_\Omega f(x, t_{\lambda_i}V_{\epsilon_0})V_{\epsilon_0}\,dx \\ &\leqslant \lim_{\lambda_i\to\infty} \frac{1}{\lambda_i}\left(\int_\Omega k(x)|x|^{-s}|V_{\epsilon_0}|^{p^*(s)}\,dx\right)^{\frac{p-1}{p-p^*(s)}} = 0, \end{aligned}$$

a contradiction with the condition (f.5) and the choice of V_{ϵ_0}. Finally we observe that

$$\sup_{t\geqslant 0} \Psi_\lambda(t) \leqslant \frac{1}{p}t^p - \frac{1}{p^*(s)}t^{p^*(s)} \int_\Omega k(x)\frac{|V_{\epsilon_0}|^{p^*(s)}}{|x|^s}\,dx,$$

and consequently, we deduce (4.4) from (4.6) and the results follow. □

References

[1] R. Dautray, J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology. Physical Origins and Classical Methods, Springer-Verlag, Berlin, 1990.

[2] J. Garcia Azorero, I. Peral, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.

[3] N. Ghoussoub, C. Yuan, Multiple solutions for quasilinear PDEs involving critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352 (2000) 5703–5743.

[4] B. Xuan, The solvability of quasilinear Brezis–Nirenberg-type problems with singular weights, Nonlinear Anal. 62 (2005) 703–725.

[5] D.G. Costa, O.H. Miyagaki, On a class of critical elliptic equations of Caffarelli–Kohn–Nirenberg type, in: Progr. Nonlinear Differential Equations Appl., vol. 66, 2005, pp. 207–220.

[6] B. Abdellaoui, V. Felli, I. Peral, Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian, Boll. Unione Mat. Ital. Sez. B 8 (2006) 445–484.

[7] D. Kang, G. Li, S. Peng, On the quasilinear elliptic problems with critical Sobolev–Hardy exponents and Hardy terms, Nonlinear Anal. 68 (2008) 1973–1985.

[8] Y. Deng, L. Jin, On symmetric solutions of a singular elliptic equation with critical Sobolev–Hardy exponent, J. Math. Anal. Appl. 329 (2007) 603–616.

[9] Z. Deng, Y. Huang, On G-symmetric solutions of critical elliptic equations of Caffarelli–Kohn–Nirenberg type, Nonlinear Anal. Real World Appl. 12 (2011) 1089–1102.

[10] G. Bianchi, J. Chabrowski, A. Szulkin, On symmetric solutions of an elliptic equations with a nonlinearity involving critical Sobolev exponent, Nonlinear Anal. 25 (1995) 41–59.

[11] T. Bartsch, M. Willem, Infinitely many Non-Radial Solutions of an Euclidean Scalar Field Equation, Mathematisches Institut/Universitat Heidelberg, 1992.

[12] J. Chabrowski, On the existence of G-symmetric solutions, Rend. Circ. Mat. Palermo 41 (1992) 413–440.

[13] R. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–39.
[14] M. Willem, Minimax Theorems, Birkhäuser, Berlin, 1996.
[15] D. Cohn, Measure Theory, Birkhäuser, 1993.
[16] P.L. Lions, The concentration–compactness principle in the calculus of variations, in: The Limit Case. Part 1, Rev. Mat. Iberoamericana 1 (1985) 145–201; The Limit Case. Part 2, Rev. Mat. Iberoamericana 2 (1985) 45–121.
[17] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.
[18] H. Rabinowitz, Methods in Critical Point Theory with Applications to Differential Equations, CBMS, Amer. Math. Soc., 1986.